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The buckling of thin viscous jets

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Thin viscous jets are considered as they slowly fall, in a state of near-neutral buoyancy, through a liquid. An equation is derived which describes the path of the jet. A small perturbation analysis of nearly vertical jets is carried out, and shows that they are necessarily unstable and will eventually deviate significantly from the vertical. Numerical integration of the nonlinear equation describes the nature of this deviation. These results model some experimental observations made by Taylor (1969).

1. Introduction

It is a common observation that thin jets of syrup, falling onto a horizontal surface, become unstable a short distance above the surface, typically spiralling in a fairly regular fashion. Taylor (1969) has suggested that this phenomenon is analogous to Euler buckling, observing that the instability is seen as soon as the jet starts to slow down (thicken), at which point that part of the axial stress associated explicitly with the velocity gradients changes from one of tension to one of compression. An equally striking instability was observed by Taylor for thin jets of glycerine falling through brine solutions of density comparable with that of the jets (i.e. the jets were close to neutral buoyancy). The instability, in this case, often manifested itself some distance from the bottom of the containing vessel, and in several cases was so violent that the jet reversed its direction, meandering upwards a considerable distance before continuing its net downward motion. Although in this case it is difficult to perceive any significant variation in jet thickness from Taylor's photographs, he suggests that the mechanism is the same as that which causes spiralling in the more familiar situation.

One of the most interesting of Taylor's photographs of glycerine in brine shows an apparent instability developing in the form of an oscillation (in space) of, initially, increasing amplitude. But instead of becoming catastrophic, and causing excessive deviations from the vertical, the oscillation dies out and the jet ultimately continues downwards in a straight line.

The suggestion by Taylor that these phenomena may be identified with Euler buckling is an interesting one, but perhaps it raises more questions than it answers. It is certainly true that there are analogies between the equations of elasticity and those of inertialess viscous flows. But consideration of the complete specification of a physical problem, including the kinematics, boundary conditions, etc., shows that the differences are as compelling, more often than not,

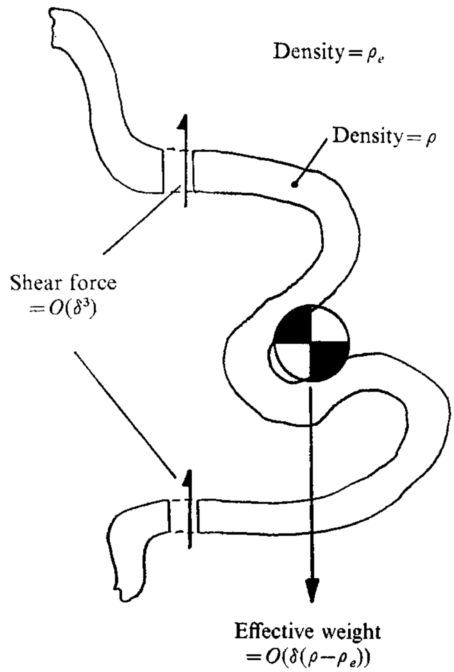

FIGURE 1. Jet supported by shear.

as the similarities. As an example we may anticipate one of the conclusions of the subsequent analysis: for an elastica, the bending moment (which plays a fundamental role in equilibrium) is proportional to the curvature; for a viscous jet, on the other hand, the bending moment depends on the *rate of change* of curvature.

The purpose of the present paper is to achieve some understanding of the glycerine-in-brine instability by consideration of a simple, rational, mathematical model. We can feel confident that such a model contains the right ingredients if it duplicates all the qualitative features of the experiment, without exhibiting any extraneous ones. In this regard, we are not completely successful. The central result is a fifth-order nonlinear differential equation which describes the location of the jet centre-line. The solutions of this equation are unstable, in the sense that a small disturbance applied to a vertical jet will eventually cause significant deviations from the vertical. Moreover, numerical integration of the equation shows that it is capable of describing meandering to a certain extent. Both of these features are revealed by Taylor's experiments. However, there is evidence that the mathematical jets will eventually spiral towards some fixed point rather than fall indefinitely, on average; and no *stable* solutions are obtained, thus leaving Taylor's single stable jet unexplained. Thus the present work should only be regarded as an exploratory first attempt at the problem, and in the conclusion suggestions are offered as to what improvements may be possible.

The fundamental assumption that we adopt, in order to make the analysis tractable, is that the jet is thin compared with a length scale on which significant flow changes occur. Furthermore, only two-dimensional jets are considered, in the belief that they should be capable of modelling three-dimensional jets that

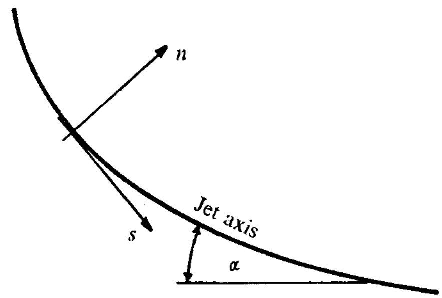

FIGURE 2. Co-ordinate system.

move in a single plane. Now in a jet of thickness $O(\delta)$, $\delta \ll 1$, it turns out that the shear stress is $O(\delta^2)$. This alone must be capable of supporting the effective weight of a unit length of jet if the model is to be capable of describing the gross meanderings apparent in Taylor's photographs (figure 1). From this it follows that the difference between the density of the jet and that of the surrounding fluid must be $O(\delta^2)$. This quantifies the assumption that the jet is close to neutral buoyancy.

The Reynolds number is assumed to be small, but it is of interest to incorporate inertial effects in a non-trivial manner, and this can be achieved by assuming that the Reynolds number is $O(\delta^2)$.

In that part of the discussion that is strictly rational, the drag imparted by the bath liquid (brine, in the experiments) is ignored. Then the only effect of the bath is to apply a hydrostatic pressure to the jet. However, it is of interest to get some idea of the role that drag would play, so that it is introduced in a *non-rational* way, in the sense that it is not deduced in a systematic manner from the fundamental equations of motion applied to the bath liquid. It must be emphasized that, if the drag is neglected, no irrationalities mar the analysis, other than those implicitly associated with formal mathematical manipulations.

The analysis starts in §2 with a derivation of the equation governing the jet path, by formal expansions in powers of δ. Substantial algebra is avoided by considering the integrated form of the equations at the appropriate point. Section 3 contains a description of a nearly vertical jet, for which the equation can be linearized. Emphasis is on the stability question. In §4 features of the nonlinear equation are described numerically and analytically, and finally, in §5, the results are summarized.

2. Derivation of the jet equation

The co-ordinate system that we shall use to describe the jet is shown in figure 2. s is distance measured along the axis of the jet, whilst n is measured perpendicular to this axis. u and v are the associated velocity components, and α is the inclination of the jet to the horizontal. In terms of this curvilinear co-ordinate system, the equations for an incompressible viscous fluid may be written in the form†

$$(\partial u/\partial s)+[\partial(hv)/\partial n] = 0, \tag{2.1a}$$

† There are countless sources for these equations; the author used Love (1944, p. 89) and Van Dyke (1969).

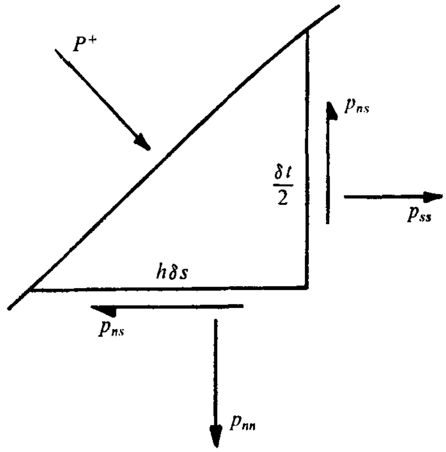

FIGURE 3. Equilibrium at jet boundary.

$$\rho\left[u\frac{\partial u}{\partial s}+v\frac{\partial}{\partial n}(hu)\right] = \rho gh\sin\alpha+\frac{\partial p_{ss}}{\partial s}+\frac{\partial}{\partial n}(hp_{ns})+p_{ns}\frac{\partial h}{\partial n}, \tag{2.1b}$$

$$\rho\left[u\frac{\partial v}{\partial s}+hv\frac{\partial v}{\partial n}-Ku^2\right] = -\rho gh\cos\alpha+\frac{\partial}{\partial n}(hp_{nn})+\frac{\partial p_{ns}}{\partial s}-p_{ss}\frac{\partial h}{\partial n}, \tag{2.1c}$$

where h is related to the curvature $K(s)$ of the jet axis by

$$h = 1+nK(s), \quad K(s) = \alpha'(s)$$

and the components of the stress tensor are related to the velocity components by means of the constitutive relations

$$p_{ss} = -p+2\mu\left(\frac{1}{h}\frac{\partial u}{\partial s}+\frac{v}{h}\frac{\partial h}{\partial n}\right), \tag{2.2a}$$

$$p_{nn} = -p+2\mu(\partial v/\partial n), \tag{2.2b}$$

$$p_{ns} = \mu\left[\frac{1}{h}\frac{\partial v}{\partial s}+h\frac{\partial}{\partial n}\left(\frac{u}{h}\right)\right]. \tag{2.2c}$$

The momentum equations, when written in terms of the velocity components, are

$$\rho\left[u\frac{\partial u}{\partial s}+v\frac{\partial}{\partial n}(hu)\right] = -\frac{\partial p}{\partial s}+\rho gh\sin\alpha+\mu h\frac{\partial}{\partial n}\left\{\frac{1}{h}\left[\frac{\partial}{\partial n}(hu)-\frac{\partial v}{\partial s}\right]\right\}, \tag{2.3a}$$

$$\rho\left[u\frac{\partial v}{\partial s}+hv\frac{\partial v}{\partial n}-Ku^2\right] = -h\frac{\partial p}{\partial n}-\rho gh\cos\alpha-\mu\frac{\partial}{\partial s}\left\{\frac{1}{h}\left[\frac{\partial}{\partial n}(hu)-\frac{\partial v}{\partial s}\right]\right\}. \tag{2.3b}$$

These equations have to be solved subject to certain boundary conditions at the edges of the jet $n = \pm\frac{1}{2}t$, where $t(s)$ is the jet thickness. The outer liquid is assumed, for the present, to be capable of imposing only a normal pressure at the boundary, so that equilibrium of an infinitesimal triangle at $n = \frac{1}{2}t$ (figure 3) then implies that

$$\left.\begin{aligned} p_{nn}(1+\tfrac{1}{2}Kt)-\tfrac{1}{2}t'p_{ns}-\rho v^2(1+\tfrac{1}{2}Kt)+\tfrac{1}{2}\rho uvt' &= -(1+\tfrac{1}{2}Kt)P^+,\\ p_{ns}(1+\tfrac{1}{2}Kt)-\tfrac{1}{2}t'p_{ss}+\tfrac{1}{2}\rho u^2t'-\rho uv(1+\tfrac{1}{2}Kt) &= \tfrac{1}{2}t'P^+ \end{aligned}\right\} \tag{2.4a}$$

at $n = \frac{1}{2}t$, and similarly, equilibrium at $n = -\frac{1}{2}t$ leads to

$$\left.\begin{aligned} p_{nn}(1-\tfrac{1}{2}Kt)+\tfrac{1}{2}t'p_{ns}-\rho v^2(1-\tfrac{1}{2}Kt)-\tfrac{1}{2}\rho uvt' &= -(1-\tfrac{1}{2}Kt)\,P^-, \\ p_{ns}(1-\tfrac{1}{2}Kt)+\tfrac{1}{2}t'p_{ss}-\tfrac{1}{2}\rho u^2t'-\rho uv(1-\tfrac{1}{2}Kt) &= -\tfrac{1}{2}t'P^- \end{aligned}\right\} \tag{2.4b}$$

at $n = -\frac{1}{2}t$. Here, $P^{\pm}$ denote the pressure in the outer liquid at $n = \pm\frac{1}{2}t$.

The remaining boundary conditions are the tangency conditions

$$hv = \pm\tfrac{1}{2}t'u \quad \text{at} \quad n = \pm\tfrac{1}{2}t. \tag{2.5}$$

We now want to relate certain integrated quantities such as the axial force, bending moment, etc. To do this, equations (2.1) are integrated across the thickness of the jet, mean quantities at the same time being defined by

$$\overline{q}(s) = \frac{1}{t}\int_{-\frac{1}{2}t}^{\frac{1}{2}t} dn\, q(n,s).$$

Thus the integrated continuity equation becomes, after use of (2.5),

$$d(t\overline{u})/ds = 0. \tag{2.6}$$

In a similar way, the integrated momentum equations, with the aid of (2.4), are

$$\rho\frac{d}{ds}(t\overline{u^2})+\rho Kt\overline{uv} = \rho gt\sin\alpha+\frac{d}{ds}(t\overline{p_{ss}})+Kt\overline{p_{ns}}+\tfrac{1}{2}t'(P^++P^-), \tag{2.7}$$

representing an axial balance, and

$$\rho\frac{d}{ds}(t\overline{uv})-\rho Kt\overline{u^2} = -\rho gt\cos\alpha+\frac{d}{ds}(t\overline{p_{ns}})-Kt\overline{p_{ss}}+(P^--P^+)-K\tfrac{1}{2}t\,(P^++P^-), \tag{2.8}$$

representing a normal balance.

The origin of the various terms in these equations is quite transparent. In particular, the external pressure provides an axial thrust due to the varying jet thickness, and a sideways thrust partly generated by the jump in pressure across the jet and partly by the curvature. It should be noted that (2.6)–(2.8) are exact.

An exact equation representing the equilibrium of moments can be obtained by multiplying the s-momentum equation by n and integrating across the jet, so that

$$\rho\frac{dI}{ds}-\rho t\overline{uv} = \rho g\sin\alpha\frac{Kt^3}{2}+\frac{dM}{ds}-t\overline{p_{ns}}+\frac{tt'}{4}(P^+-P^-), \tag{2.9}$$

where M and I are 'bending moments', defined by

$$M = \int_{-\frac{1}{2}t}^{\frac{1}{2}t} dn\, np_{ss}, \quad I = \int_{-\frac{1}{2}t}^{\frac{1}{2}t} dn\, nu^2.$$

Another useful relationship is obtained by integrating (2.2a), whence

$$t\overline{p_{ss}} = -t\overline{p}-2\mu(v^+-v^-). \tag{2.10}$$

The integrated equations derived above are insufficient to solve the problem of course, since they contain too many unknowns. Consequently they must be supplemented by closure conditions. For example, in the related problem of the elastica, the bending moment is classically related to the curvature by assuming that plane sections remain plane. Here we shall derive closure conditions under the assumption that the jet is very thin, is in a state of near-neutral buoyancy, is nearly inertialess and is immersed in a bath of low viscosity fluid.

It is clear that if the jet's density is equal to that of the bath liquid, and the latter is inviscid, then the equations admit a straight jet of constant thickness travelling at an arbitrary uniform velocity. Moreover, if the jet is very thin, it need not be straight. Indeed, to leading order, the equations admit an *arbitrary* path. Such a solution provides the leading term in an expansion in powers of the jet thickness. The path is subsequently determined by a solubility condition on higher order terms in the expansion.

Define non-dimensional quantities with the aid of a characteristic length L, a characteristic speed U, a characteristic stress $\mu(U/L)$, and a characteristic density ρ_e equal to that of the bath liquid. The governing equations are then

$$\partial\tilde{u}/\partial S + (\partial(h\tilde{v})/\partial\tilde{n}) = 0, \tag{2.11a}$$

$$\frac{\rho}{\rho_e} Re\left[\tilde{u}\frac{\partial\tilde{u}}{\partial S} + \tilde{v}\frac{\partial}{\partial\tilde{n}}(h\tilde{u})\right] = -\frac{\partial\tilde{p}}{\partial S} + \frac{\rho}{\rho_e} Gh\sin\alpha h + \frac{\partial}{\partial\tilde{n}}\left\{\frac{1}{h}\left[\frac{\partial}{\partial\tilde{n}}(h\tilde{u}) - \frac{\partial\tilde{v}}{\partial S}\right]\right\}, \tag{2.11b}$$

$$\frac{\rho}{\rho_e} Re\left[\tilde{u}\frac{\partial\tilde{v}}{\partial S} + h\tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{n}} - \tilde{K}\tilde{u}^2\right] = -h\frac{\partial\tilde{p}}{\partial\tilde{n}} - \frac{\rho}{\rho_e} Gh\cos\alpha - \frac{\partial}{\partial S}\left\{\frac{1}{h}\left[\frac{\partial}{\partial\tilde{n}}(h\tilde{u}) - \frac{\partial\tilde{v}}{\partial S}\right]\right\}, \tag{2.11c}$$

where $\quad Re = \rho_e UL/\mu, \quad G = \rho_e gL^2/\mu U, \quad s = LS$

and quantities with a tilde are non-dimensional.

The thickness of the jet is $O(\delta)$, $\delta \ll 1$, and then the expansion for a slender jet is based on the assumptions

$$\rho/\rho_e = 1 + \rho^*\delta^2, \quad \rho^* = O(1), \quad Re = \delta^2 Re_2 + \ldots.$$

As explained in the introduction, this is necessary if our model is to have any hope of duplicating the meanderings revealed by Taylor's photographs.

All variables are expanded as a power series in δ,

$$\left.\begin{aligned} \tilde{t} &= \delta t_1 + \delta^2 t_2 + \ldots, & \tilde{u} &= u_0 + \delta u_1 + \ldots, \\ \tilde{v} &= \delta v_1 + \delta^2 v_2 + \ldots, & \alpha &= \alpha_0 + \delta\alpha_1 + \ldots, \\ \tilde{K} &= K_0 + \delta K_1 + \ldots, & \tilde{p} &= p_0 + \delta p_1 + \ldots, & &\text{etc.} \end{aligned}\right\} \tag{2.12}$$

where the variables on the right side are functions of S and the scaled normal variable N, related to $\tilde{n}$ by $\tilde{n} = \delta N$.

The above expansions are substituted into (2.11). The procedure for solving the resulting system of linear equations is to first find u_j from the s-momentum equation, and then the other two equations determine v_{j+1} and p_j. At each stage (particular value of j) the solutions contain undetermined functions of S. Some of these are related by conditions at the jet boundaries, but complete determination of the solution to any particular order does not come until several of the higher order terms are considered. K_0, for example, which is of primary interest, is not determined by this procedure without consideration of u_3, v_4 and p_3. The role of the integrated equations (2.6)–(2.10) is to bypass a substantial amount of algebra by permitting the calculation of K_0 without explicit evaluation of u_3, v_4 and p_3.

The leading solution is very simple:

$$\left.\begin{aligned} u_0 &= \text{constant}, \quad K_0 = K_0(S), \quad v_1 = 0, \\ p_0 &= P_{c_0}(S) = \tfrac{1}{2}(P^+ + P^-)_0, \quad t_1 = \text{constant}, \end{aligned}\right\} \tag{2.13}$$

where we have introduced $P_c(S) = \frac{1}{2}(P^+ + P^-)$, the pressure in the outer liquid measured at the axis of the jet.†

Note that, from hydrostatics,

$$P^- - P_c = P_c - P^+ = \tfrac{1}{2}\rho_e g t \cos\alpha. \tag{2.14}$$

The important thing to notice about the leading solution (2.13) is that K_0 (and therefore α_0) is not determined. Restrictions on the path only arise by consideration of higher order terms, so that the ultimate description will be nonlinear.

The general solution of the equations for u_1, v_2 and p_1 is

$$u_1 = u_{11}(S)\,N + u_{10}(S), \tag{2.15a}$$

$$v_2 = -\tfrac{1}{2}u'_{11}N^2 - u'_{10}N + v_{20}(S), \tag{2.15b}$$

$$p_1 = [-G(\cos\alpha)_0 - (u'_{11} + u_0 K'_0)]\,N + p_{10}(S) \tag{2.15c}$$

and applying the boundary conditions leads to

$$u_1 = K_0 u_0 N + u_{10}, \quad \text{where} \quad u_0 t'_2 + u'_{10} t_1 = 0, \tag{2.16a}$$

$$v_2 = -\tfrac{1}{2}u_0 K'_0 N^2 - u'_{10}N + \tfrac{1}{8}t_1^2 K'_0 u_0, \tag{2.16b}$$

$$p_1 = [-G(\cos\alpha)_0 - 2u_0 K'_0]\,N + (P_{c_1} - 2u'_{10}). \tag{2.16c}$$

Nothing else is determined at this stage of the calculation, but turning to the next set of equations (those for u_2, v_3 and p_2) and boundary conditions leads, after some calculation, to the restraint

$$u'_{10}K_0 = 0. \tag{2.17}$$

K_0 will be assumed *not* to vanish, so that we find

$$u_2 = (K_0 u_{10} + K_1 u_0)\,N + u_{20}, \quad \text{where} \quad t_1 u'_{20} + u_0 t'_3 = 0, \tag{2.18a}$$

$$\begin{aligned} v_3 = {}& \tfrac{1}{2}u_0 K_0 K'_0 N^3 - \tfrac{1}{2}(K'_0 u_{10} + K'_1 u_0)\,N^2 \\ & - (u'_{20} + \tfrac{1}{8}t_1^2 K_0 K'_0 u_0)\,N + \tfrac{1}{8}t_1^2(K'_0 u_{10} + K'_1 u_0) + \tfrac{1}{4}t_1 t_2 u_0 K'_0, \end{aligned} \tag{2.18b}$$

$$p_2 = u_0 K_0 K'_0 N^2 + [-G(\cos\alpha)_1 - 2(u_0 K'_1 + u_{10}K'_0)]\,N + p_{20}, \tag{2.18c}$$

where
$$p_{20} = P_{c_2} + \tfrac{1}{4}t_1^2 u_0 K_0 K'_0 - 2u'_{20}$$
and u_{10} is a constant.

We could continue in this way, solving next for u_3, v_4 and p_3, but there are two reasons why we do not want to do this. For one thing, the amount of algebra would be quite considerable and yet most of the information this would yield is not of interest. It is not u_3, v_4 and p_3 that we want to find, but rather the condition

† It should not be thought that expanding this as a power series in δ implies that the motion of the jet disturbs the outer fluid. The pressure in the bath is always defined by the law of hydrostatistics. It is the *path* of the jet that depends on δ, and thus causes the dependence of P_c.

on K_0 that their solution implies.† In addition we want to introduce an $O(\delta^3)$ drag into the formulation without getting involved in the precise physical mechanisms which generate this drag. The simplest way to introduce such an effect is to add a drag term to the equation which represents the axial force balance, equation (2.7). This will be effective provided that we can switch to the integrated form of the equations at this point. It turns out that, indeed, we now have enough information to close them to the required order. Such simplicity is not achieved without a price however. The external pressure must still be specified, and the only possible choice is the hydrostatic law. In other words we *model* the bath liquid by a medium that imparts a hydrostatic pressure to the jet and a constant drag per unit length. These must surely be the fundamental ingredients of a real flow, but there would be additional (and very complicated) terms, so that the correlation can, at best, be qualitative. It must be emphasized, however, that such a compromise is not essential to our analysis. For no compromise is necessary if the drag is zero, corresponding to an inviscid (relative to the jet), inertialess bath fluid.

The equations governing the averaged quantities have already been written down [(2.6)–(2.10)], and will be used as they stand with the exception of (2.7), which is replaced by (in non-dimensional form)

$$\frac{\rho}{\rho_e} Re \frac{d}{dS}(\widetilde{\tilde{t}u^2}) + \frac{\rho}{\rho_e} Re \tilde{K} \widetilde{\tilde{t}uv} = \frac{\rho}{\rho_e} G\tilde{t} \sin\alpha + \frac{d}{dS}(\tilde{t}\widetilde{p_{ss}}) + \tilde{K}\tilde{t}\widetilde{p_{ns}} + \tilde{t}' \tilde{P}_c - \delta^3 \tilde{D}, \quad (2.19)$$

where $\tilde{D}$ is a non-dimensional constant drag. $\tilde{D}$ will have the same sign as u_0.

These integrated equations have to be closed, and this is easily done using the solutions (2.16) and (2.18). Thus the axial stress is given by

$$\tilde{p}_{ss} = -p_0 + \delta[(G(\cos\alpha)_0 + 4u_0 K_0')N - P_{c_1}] + O(\delta^2),$$

from which follows

$$\tilde{M} = \tfrac{1}{12}\delta^3[G(\cos\alpha)_0 + 4u_0 K_0'] t_1^3 + \dots. \quad (2.20)$$

There are two contributions to the bending moment, just as there are in elastic-beam theory. The load, in this case the weight, makes a contribution and so does the curvature. However, since u_1 does not vary with S if the curvature is constant, the moment is not proportional to the curvature, but instead depends on the *derivative* of the curvature. Clearly there is no real analogy with elastic-beam theory, and it is probably not helpful to think of the instability in terms of Euler buckling.

The remaining closure conditions are

$$(\tilde{v}^+ - \tilde{v}^-) = \delta^3 u_0 t_3' + O(\delta^4) \quad (2.21)$$

and

$$\tilde{\tilde{p}} = p_0 + \delta P_{c_1} + \delta^2(p_{20} + \tfrac{1}{12} t_1^2 u_0 K_0 K_0') + O(\delta^3). \quad (2.22)$$

In addition the mean axial stress turns out to be a useful quantity:

$$-\widetilde{\tilde{t}p_{ss}} = \delta p_0 t_1 + \delta^2(t_1 P_{c_1} + p_0 t_2) + \delta^3(t_1 P_{c_2} + t_2 P_{c_1} + t_3 p_0 + \tfrac{1}{3} t_1^3 u_0 K_0 K_0' + 4u_0 t_3') + O(\delta^4). \quad (2.23)$$

† We can be confident that there will be such a condition, since it is at this stage that ρ^*, which is a measure of the density difference, first plays a role.

The integrated equations can now be solved. Thus the two force equilibrium equations yield

$$0 = \rho^* G(\sin\alpha)_0 t_1 + G[(\sin\alpha)_0 t_3 + (\sin\alpha)_1 t_2 + (\sin\alpha)_2 t_1] + [\widetilde{\tilde{t}\overline{p}_{ss}}]_3' + K_0 t_1(\overline{p_{ns}})_2 + t_3' P_{c_0} - \tilde{D}, \quad (2.24a)$$

$$-Re_2 K_0 t_1 u_0^2 = -\rho^* G(\cos\alpha)_0 t_1 + t_1[(\overline{p_{ns}})_2]' - K_0[\widetilde{\tilde{t}\overline{p}_{ss}}]_3 - K_0[t_1 P_{c_2} + t_2 P_{c_1} + t_3 P_{c_0}]. \quad (2.24b)$$

Also, the moment equation, together with the result (2.20) for M_3, implies that

$$(\overline{p_{ns}})_2 = \tfrac{1}{3} t_1^2 u_0 K_0'', \quad (2.25)$$

where we have recalled that K and α are related, namely $K = \alpha'$. Eliminating $(\overline{p_{ns}})_2$ and $[\widetilde{\tilde{t}\overline{p}_{ss}}]_3$ from (2.24) and (2.25) then leads to the following equation for the inclination α_0:

$$0 = \Omega\left[2\sin\alpha_0 + \cos\alpha_0 \frac{\alpha_0''}{(\alpha_0')^2}\right] + \alpha_0'\alpha_0''' + \left(\frac{\alpha_0^{\mathrm{iv}}}{\alpha_0'}\right)' - D^*, \quad (2.26)$$

where $$\Omega \equiv 3\rho^* G/t_1^2 u_0, \quad D^* = 3\tilde{D}/t_1^3 u_0.$$

Apart from the drag, Ω is the single non-dimensional parameter of the problem, and in physical variables is proportional to $(\rho - \rho_e) L^4 g/\mu U t^2$. If $D^* = 0$, Ω can be eliminated from (2.26) by a simple scaling. Note that D^* is non-negative.

3. Analysis of a nearly vertical jet

The basic features of nearly neutrally buoyant jets as revealed by Taylor's experiments is that they are unstable, in general, and do not remain vertical. In this section we examine the stability characteristics of our model equation (2.26) by considering jets that deviate only a little from the vertical.

Write

$$\alpha_0 = \tfrac{1}{2}\pi + \epsilon\theta + \ldots \quad (\epsilon \ll 1),$$

and substitute into (2.26), retaining only leading terms for small ϵ. This yields an equation for θ, namely

$$\theta'\theta^{\mathrm{v}} - \theta''\theta^{\mathrm{iv}} - \Omega\theta\theta'' + (2\Omega - D^*)\theta'^2 = 0. \quad (3.1)$$

This equation can be integrated once to yield a linear fourth-order equation containing an arbitrary constant of integration. To relate this constant to the flow variables, we may use the expression for $[\widetilde{\tilde{t}\overline{p}_{ss}}]_2$ in (2.23), together with (2.24) and (2.25), to obtain an expression for t_3', namely

$$4u_0 t_3' = -\frac{t_1^3}{3} u_0 K_0 K_0' + \rho^* G \frac{(\cos\alpha)_0}{K_0} t_1 - \frac{t_1^3}{3} u_0 \frac{K_0'''}{K_0} - Re_2 t_1 u_0^2. \quad (3.2)$$

Differentiating this expression, and using the governing equation for K_0, yields

$$(12/t_1^3)\, t_3'' = \Omega \sin\alpha_0 - D^* - K_0'^2, \quad (3.3)$$

where, for a nearly vertical jet, the right-hand side can be replaced by $\Omega - D^*$. Since t_3' is proportional to u_{20}', we conclude that

$$u_{20}' = -\tfrac{1}{12}u_0 t_1^2 (\Omega - D^*) S + [u_{20}(1) - u_{20}(0)] + \tfrac{1}{24}u_0 t_1^2 (\Omega - D^*) \tag{3.4}$$

for a nearly vertical jet, with a similar expression for t_3'. Substituting back into (3.2) then gives

$$\theta^{\mathrm{iv}} + \theta' \left[(\Omega - D^*) S - \frac{12}{u_0 t_1^2}[u_{20}(1) - u_{20}(0)] - \tfrac{1}{2}(\Omega - D^*) + 3Re_2 \frac{u_0}{t_1^2} \right] + \Omega\theta = 0, \tag{3.5}$$

and this is the first integral of (3.1).

The analysis in this section consists of a discussion of the solution of (3.5), but consider first the velocity u_{20}. Since $u_{10}' = 0$ it is clear that u_{20} provides the first term of $\overline{u}$, the mean jet velocity, that varies with S. This variation is quadratic in S and implies that, for very large S, $|u_{20}|$ will become very large. This is not surprising, since the smallest density difference can have an appreciable effect if the associated gravitational force acts on a fluid particle for a sufficiently large time. Naturally then, the present analysis is only valid on a bounded interval, and certainly breaks down when $S = O(1/\delta)$.

u_{20}, and therefore $\overline{u}$, has a turning-point at

$$S_t = \frac{12[u_{20}(1) - u_{20}(0)]}{u_0 t_1^2(\Omega - D^*)} + \frac{1}{2}. \tag{3.6}$$

It follows that, provided u_0 is positive, then $\overline{u}$ has a maximum at this turning-point if $\Omega > D^*$, and the jet thickness is then a minimum. Thus, when the weight is greater than the drag, there is a point where the stress explicitly generated by the velocity field changes from one of tension to one of compression. This is the same situation as occurs in a jet of syrup falling through air onto a horizontal surface, and with which Taylor (1969) has identified the onset of instability. If the drag is greater than the weight, however, ($\Omega < D^*$), $\overline{u}$ has a *minimum* at the turning-point and the stress changes from compressive to tensile. This would not normally be observed in more common situations where the jet is not close to neutral buoyancy. The results obtained in this section imply that this situation is also unstable.†

Equation (3.5) is best discussed in two parts, one for positive values of Ω, the other for negative values. Consider first of all positive values of Ω and introduce new variables m and ϕ, defined by

$$(\Omega - D^*) S - \frac{12}{u_0 t_1^2}[u_{20}(1) - u_{20}(0)] - \tfrac{1}{2}(\Omega - D^*) + 3\frac{Re_2 u_0}{t_1^2} = \frac{\Omega - D^*}{\Omega^{\frac{1}{4}}} m,$$

$$\theta(S) = \phi(m),$$

so that (3.5) becomes

$$\phi^{\mathrm{iv}}(m) + \omega m \phi'(m) + \phi(m) = 0, \tag{3.7}$$

$$\omega \equiv 1 - D^*/\Omega.$$

† The dichotomy described in this paragraph can be duplicated even if the drag is zero, since the sign of Ω can be changed by changing the sign of ρ^*.

$m = 0$ is a natural origin for this equation, but it does not coincide with the turning-point (3.6). Specifically, S_0, the value of S when $m = 0$, is related to S_t by

$$S_0 = S_t - (3Re_2 u_0/t_1^2(\Omega - D^*)).$$

It is of interest to discuss the general solution of (3.7) for values of $\omega \leqslant 1$. However, some preliminary clues to the nature of the solutions that can be expected can be obtained by seeking asymptotic solutions, valid for large $|m|$, directly from (3.7). Thus if we seek an asymptotic solution of the form

$$\phi \sim |m|^{\alpha} \exp\{k|m|^{\beta}\}$$

we find that, for positive values of ω, three possibilities are

$$\phi \sim |m|^{\alpha_1} \exp(-\tfrac{3}{4}\omega^{\frac{1}{3}}|m|^{\frac{4}{3}}), \quad |m|^{\alpha_2} \exp(\tfrac{3}{8}\omega^{\frac{1}{3}}|m|^{\frac{4}{3}}) \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{3\sqrt{3}}{8}\,\omega^{\frac{1}{3}}\,|m|^{\frac{4}{3}}\right), \tag{3.8a}$$

whereas, if $\omega < 0$, the possibilities are

$$\phi \sim |m|^{\alpha_3} \exp\{\tfrac{3}{4}(-\omega)^{\frac{1}{3}}\,|m|^{\frac{4}{3}}\}, \quad |m|^{\alpha_4} \exp(-\tfrac{3}{8}(-\omega)^{\frac{1}{3}}|m|^{\frac{4}{3}}) \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{3\sqrt{3}}{8}\,(-\omega)^{\frac{1}{3}}\,|m|^{\frac{4}{3}}\right). \tag{3.8b}$$

The fourth solution is algebraic:

$$\phi \sim |m|^{-1/\omega}. \tag{3.8c}$$

Solutions of (3.7) can be found in the form

$$\phi = \int_C dt \exp\left\{-mt + \frac{t^4}{4\omega}\right\} t^{(1-\omega)/\omega}, \tag{3.9}$$

where C is any contour for which $t^{1/\omega} \exp(-mt + t^4/4\omega)$ vanishes at each end. If $\omega > 0$, suitable contours are straight lines originating at the origin and going to infinity at angles $\frac{1}{4}\pi$, $\frac{3}{4}\pi$, $\frac{5}{4}\pi$ and $\frac{7}{4}\pi$ relative to the positive real axis. Each of these four contours leads to an independent solution, so that the general solution is a linear combination of the four solutions:

$$\phi = \int_0^\infty dr \exp\left\{\pm mr - \frac{r^4}{\omega}\right\} r^{1/\omega - 1} \begin{matrix}\cos\\ \sin\end{matrix}(mr) \quad (\omega > 0). \tag{3.10}$$

It is clear that two of these solutions diverge exponentially as $m \to \infty$ and two of them decay algebraically. A unique linear combination of the two asymptotically algebraic solutions, namely

$$\int_0^\infty dr \exp\left\{-mr - \frac{r^4}{\omega}\right\} r^{1/\omega - 1}\left[\sin\left(\frac{\pi}{4\omega}\right)\cos mr - \cos\left(\frac{\pi}{4\omega}\right)\sin mr\right],$$

decays exponentially [cf. (3.8a)].

The most significant feature of these results is that the general solution for ϕ contains terms that diverge exponentially for large negative values of m, and terms that diverge exponentially for large positive values of m. The physical manifestation of this is, presumably, a significant deviation of the jet from the vertical, in agreement with the experimental observations of Taylor. This instability is in no way associated with traversal of the origin ($m = 0$), and therefore has nothing to do with S_t. *It can be initiated whether or not the velocity gradient is negative.*

The contours used above will not do if $\omega < 0$. One suitable contour traverses the entire length of the imaginary axis with the exception of a small detour around the singularity at the origin. This gives one solution from which a second independent solution can be obtained, if $1/\omega$ is not an integer, by replacing m by $-m$.† The two solutions obtained in this way are

$$\phi(m) = \int_0^\infty dr \exp\left\{\frac{r^4}{4\omega}\right\} r^{1/\omega-1} \begin{matrix}\sin\\ \cos\end{matrix}(mr) \quad (\omega < 0), \tag{3.11a}$$

where only the *finite* part of the integral is retained. Two other solutions can be obtained in a similar way by considering a contour along the real axis. This yields

$$\phi = \int_0^\infty dr \exp\left\{\pm mr + \frac{r^4}{4\omega}\right\} r^{1/\omega-1} \quad (\omega < 0), \tag{3.11b}$$

where, again, only the finite part is retained.

Three of these solutions behave algebraically as $m \to \infty$ and one diverges exponentially. However, the two linear combinations

$$-\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\pi}{2\omega}\right)\int_0^\infty dr \exp\left\{-mr + \frac{r^4}{4\omega}\right\} r^{1/\omega-1} + \int_0^\infty dr \exp\left\{\frac{r^4}{4\omega}\right\} r^{1/\omega-1} \begin{matrix}\cos\\ \sin\end{matrix}(mr) \tag{3.11c}$$

both decay exponentially as $m \to \infty$. These results are consistent with (3.8*b*).

The solutions for $\omega < 0$ predict instability despite the fact that the speed of the jet then displays a minimum. Moreover, if $\omega = 0$, so that u_{20} varies linearly with S, equation (3.5) for θ still has unstable solutions. Thus for the present problem it appears that instability cannot be associated with a particular sign for the velocity gradient, contrary to the speculations of Taylor.

The solution of (3.5) for negative values of Ω is described in terms of the variables

$$(\Omega - D^*)S - \frac{12}{u_0 t_1^2}[u_{20}(1) - u_{20}(0)] - \tfrac{1}{2}(\Omega - D^*) + 3\frac{Re_2 u_0}{t_1^2} = \frac{\Omega - D^*}{(-\Omega)^{\frac{1}{4}}} l,$$

$$\theta(S) = \chi(l),$$

so that

$$\chi^{\text{iv}} - \omega l \chi' - \chi = 0, \tag{3.12}$$

where ω is positive. Solutions are

$$\chi = \int_0^\infty dr \exp\left\{\pm lr - \frac{r^4}{4\omega}\right\} r^{1/\omega-1}, \tag{3.13a}$$

$$\chi = \int_0^\infty dr \exp\left\{-\frac{r^4}{r\omega}\right\} r^{1/\omega-1} \begin{matrix}\sin\\ \cos\end{matrix}(lr), \tag{3.13b}$$

and one of these diverges exponentially, implying instability.

† The case when $1/\omega$ is a negative integer is distinguished by the fact that there is a polynomial solution. We do not discuss it since it is not of special interest, and the implications for the stability question do not differ in essence from those deduced for arbitrary negative ω.

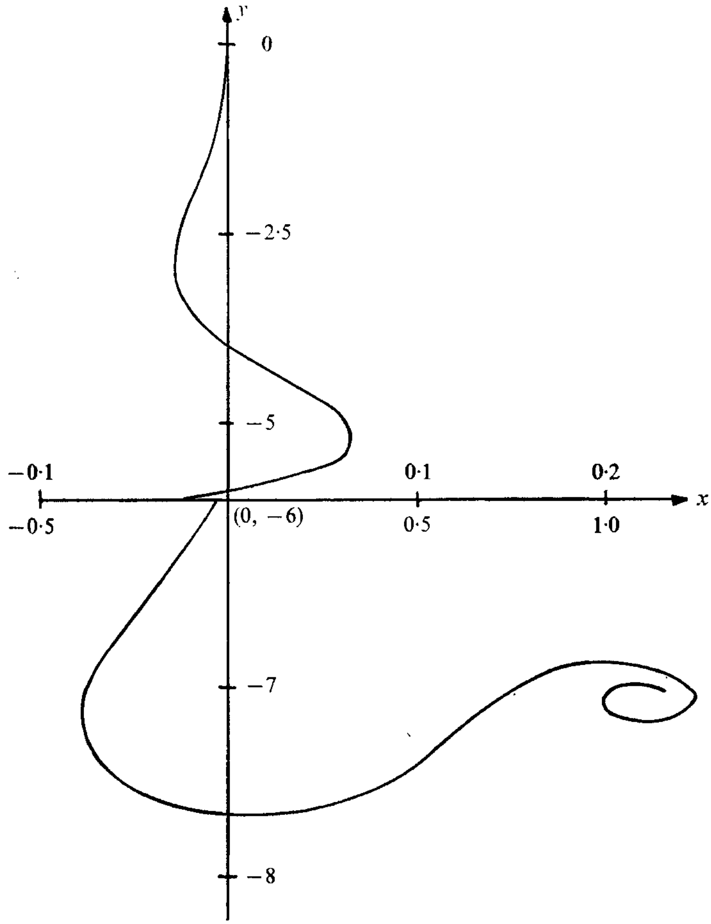

FIGURE 4. Nonlinear jet.

4. Nonlinear analysis

In §2, the equation governing the path of the jet was derived, namely (with the subscript dropped)

$$\alpha'\alpha^{\mathrm{v}} - \alpha^{\mathrm{iv}}\alpha'' + \alpha'^3\alpha''' - D^*\alpha'^2 + \Omega[2\alpha'^2 \sin\alpha + \alpha'' \cos\alpha] = 0. \tag{4.1}$$

Numerical integration of this equation as an initial-value problem complements the conclusions of §3 that nearly vertical jets are unstable, in that it describes the deviation beyond the point where the linearized analysis breaks down. It does not replace the linear analysis since the disturbances introduced are, of necessity, finite. A typical result of such a computation is shown in figure 4 ($\Omega = 1, D^* = 0$). x and y are Cartesian co-ordinates with the gravity force in the negative-y direction; note the scale change. Substantial deviations from the vertical, including reversal, were observed by Taylor.

All the numerical computations reveal a common asymptotic behaviour in which the jet eventually turns along an ever-tightening spiral path. It cannot be concluded that this is the only possible asymptotic behaviour, but it is of interest to see whether it can be described analytically.

Numerical integration suggests that, as $S \to \infty$, there is a balance

$$\alpha^{\mathrm{v}} + \alpha'^2 \alpha''' \sim 0. \tag{4.2}$$

This in turn suggests that α' has the asymptotic form

$$\alpha' \sim AS + B + \ldots, \tag{4.3}$$

for then (4.2) is an equation for α''', with parabolic cylinder functions as solutions, so that

$$\alpha''' \sim \frac{C}{S^{\frac{1}{2}}} \exp\left\{\frac{i}{2A}(AS+B)^2\right\} + \frac{D}{S^{\frac{1}{2}}} \exp\left\{-\frac{i}{2A}(AS+B)^2\right\} + \ldots;$$

and integration leads to

$$\alpha' \sim AS + B - \frac{C}{A^2}\frac{1}{S^{\frac{5}{2}}} \exp\left\{\frac{i}{2A}(AS+B)^2\right\} - \frac{D}{A^2}\frac{1}{S^{\frac{5}{2}}} \exp\left\{-\frac{i}{2A}(AS+B)^2\right\} + \ldots, \tag{4.4}$$

consistent with (4.3). It is easily verified that the terms omitted from (4.2) are smaller than those retained. Note also that (4.4) implies that the expansion for α contains five arbitrary constants, so that it constitutes a general solution.

The path of the jet in this asymptotic limit is

$$x \sim x_0 + \frac{\sin\left(\frac{1}{2}AS^2 + BS + E\right)}{AS} + \ldots, \quad y \sim y_0 + \frac{\cos\left(\frac{1}{2}AS^2 + BS + E\right)}{AS} + \ldots,$$

where $\alpha \sim \frac{1}{2}AS^2 + BS + E$ and x_0 and y_0 are constants. This is a spiral whose radius $\sim 1/S$, consistent with the numerical results.

A spiral of this kind is not seen experimentally. Although there is a strong tendency, given the right conditions, for a gently oscillating jet suddenly to wrap around in a tight loop, it does not repeat the circuit but instead turns, quite sharply, out of the plane of the loop (see especially figure 9 in Taylor's paper). It is possible, then, that the spiral is a creature of the two-dimensional analysis. The failure of the path to display any long range drift under the influence of gravity is presumably a result of the non-uniformity mentioned in the paragraph preceding equation (3.6).

5. Concluding remarks

The work described in this paper was motivated by experiments of Taylor (1969), in which thin, slowly moving jets of glycerine (viscid) were observed as they travelled vertically downwards through brine solutions (relatively inviscid) of density only slightly less than that of the glycerine. These experiments reveal that such jets can be unstable and deviate significantly from the vertical. Taylor has suggested that the instability is analogous to Euler buckling.

The present work has attempted to model these results by consideration of thin two-dimensional jets for which the density difference is $O[(\text{thickness})^2]$. The analysis of § 2 shows that such jets are undoubtedly unstable. However, the instability seems to have little to do with the sign of the velocity gradient. Thus a jet which is in a state of tension will rapidly deviate from the vertical without waiting for the stress to become compressive. Furthermore the equations governing the shape of the jet have little in common with the equation of an elastica. Consequently there appears to be no evidence that this phenomenon is related to Euler buckling.†

The present results do not provide a stability criterion in the sense of a critical value of some parameter necessary for instability to occur. However the argument based on figure 1 implies that if $\rho/\rho_e - 1$ is $O(\delta)$, say, then $O(1)$ deviations from the vertical cannot occur. Analysis of this case confirms such a point of view in that the deviation from the vertical is $O(\delta)$. The connexion with the present analysis is that as $\Omega \rightarrow \infty$ the domain of validity shrinks to zero (this is clear from (3.4)). Thus a greater understanding of the stability question is likely to arise from an examination of the non-uniformity that arises at large values of S. An analysis of this kind would involve the use of multiple scales.

Numerical integration of the nonlinear jet equation has described the nature of the instability. Typically, after a few relatively mild oscillations, an initially straight jet quickly tightens into a loop. This closely parallels figure 9 of Taylor's paper. However, the theoretical jet continues to circle, following a spiral path, whereas real jets turn out of the plane of the loop. This suggests that it would be of interest to examine the three-dimensional problem.

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† It is worth noting that Taylor describes another experiment in his 1969 paper in which threads of a very viscous liquid, floating on mercury, are buckled. Explicit comparison is made in figure 12 of that work with the third mode of buckling of an elastica. In an analysis related to the present one, A. Nachman and the author have derived equations which describe the motion of a thin viscous thread. Although shapes that are roughly elastica-like can be generated, the equations and their solutions have little in common with those of an elastica.